Problem Sheet 13

Exercise 13.1

Let $(\Omega, \mathcal{F}, \mathbb{P}), \theta$ be a dynamical system. Prove that the following statements are equivalent:

1. $\mathbb{P}$ is ergodic (θ is ergodic);
2. Every invariant integrable function f is almost surely a constant.
3. Every invariant bounded function is almost surely a constant.

Exercise 13.2

Let T be a Feller transition operator that furthermore preserves boundedness of functions. Prove that the set of invariant probability measures for T is closed in the topology of weak convergence.

Exercise 13.3 (*Taken from Martin Hairer's 'Ergodic Theory for Stochastic PDEs').*

Let $\mathcal{X} = [0, 1]$, and define the transition probability

$$P(x, \cdot) = \begin{cases} \delta_{\frac{x}{2}} \text{ if } x > 0, \\ \delta_1 \text{ if } x = 0. \end{cases}$$

1. Prove that P cannot have an invariant measure.
2. Why does the Krylov-Bogoliubov Theorem not apply?
3. Suppose that we now alter the topology of $\mathcal{X}$ by including $\{0\}$. The Borel sets for $\mathcal{X}$ are unchanged, hence the notion of invariant measure remains unchanged too. Why, now, does the Krylov-Bogolibuov Theorem not apply?

Exercise 13.4

Let (ξ_n) be i.i.d random variables on $\mathbb{R}$ with $\mathbb{P}(\xi_n = 1) = \frac{1}{2}$ and $\mathbb{P}(\xi_n = -1) = \frac{1}{2}$. Let (x_n) be a Markov Chain on $\mathbb{R}$ given by

$$x_{n+1} = \frac{1}{1 + x_n^2} + \xi_n.$$

Show that x_n has an invariant probability measure.

Exercise 13.5

Let $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$ be a dynamical system. Show that if θ is ergodic and $f \in L^1(\Omega, \mathcal{F}, \mathbb{P})$, then

$$\lim_{n \to \infty} \frac{f(\theta^n)}{n} = 0$$

$\mathbb{P} - a.s.$.